Probability Model of Reliability on Some Distributions

May Thida Htun[1] and Win Win Ei[2]

Abstract

This paper deals with the study on reliability theory. Applications of Poisson distribution, Normal distribution, and Exponential distribution using in reliability are discussed. The Weibull failure law and reliability of systems of components are also studied. Applications of series and parallel system of components can be seen.

Key words: Reliability, Failure law, Series system, Parallel system

Introduction

Nowadays, in modern developed countries, there are Reliability Engineers who play the important role in aerospace technology as well as in various technological fields. Theory of reliability is applied to test the life time of a component or device.

Reliability has many different technical meanings. Reliability is defined as the probability of a device performing of its purpose adequate for the period intended under the given operating conditions.

When a system or unit does not perform satisfactorily, it is said to have failed and the failure is a function of time. The failure rate can be plotted against time and the failure is a function of time.

The reliability of a component equals the probability that the component does not fail during the given time or equals the probability that the component is still functioning at given time. In addition, the reliability function plays an important role in describing the failure characteristics of an item.

Reliability Function

Let X be a fixed number of components to be tested. In this case trails are assumed to be mutually exclusive and equally likely. After time t, with $X_s(t)$ surviving and $X_f(t)$ failing, where $X = X_s(t) + X_f(t)$ (Meyer, 1966) .

Reliability function, the probability of survival, is defined as

$$R(t) = \frac{X_s(t)}{X} \tag{1}$$

and the probability of failure $F(t) = 1 - R(t) = \frac{X_f(t)}{X}$. (2)

Therefore

$$\frac{d}{dt} X_f(t) = -X \frac{d}{dt} R(t), \tag{3}$$

and

$$\frac{d}{dt} X_s(t) = -\frac{d}{dt} X_f(t). \tag{4}$$

---

[1]Assistant Lecturer, Dr., Department of Mathematics, Hinthada University
[2]Associate Professor, Dr., Department of Mathematics, Hinthada University

Equation (4) shows that the rate of survival of components is the negative of the rate at which components fail. From equations (1) and (3), it follows that

$$\frac{1}{X_s(t)}\frac{d}{dt}X_f(t) = -\frac{1}{R(t)}\frac{d}{dt}R(t). \qquad (5)$$

Left side of equation (5) is the failure rate or instantaneous probability of failure per component and is denoted by λ(t). Then

$$\lambda(t) = -\frac{1}{R(t)}\frac{d}{dt}R(t). \qquad (6)$$

Therefore, $R(t) = e^{-\int_0^t \lambda(t)dt}$. (7)

From equation (7), the properties of R (t) are as follows:

(i) $0 \le R(t) \le t$,

(ii) R (0) = 1 and R (∞) = 0,

(iii) R(t) is, in general, decreasing function. If f (t) is the density of continuous random variable T and F(t) is its distribution function ,

(iv) $R(t) = 1 - f(t) = 1 - \int_0^t f(\tau)d\tau = \int_t^\infty f(\tau)d\tau.$ (8)

Failure rate λ(t) and Hazard rate h(t)

The rate at which failures occur in the time interval (t_1, t_2) is called the failure rate and is expressed as the conditional probability that failures occur in (t_1, t_2) given that failures have not occurred prior to t_1. Then

$$\lambda(t) = \frac{\int_{t_1}^{t_2} f(t)dt}{(t_2 - t_1)\int_{t_1}^{\infty} f(t)dt} = \frac{R(t_1) - R(t_2)}{\Delta t\, R(t)}, \qquad (9)$$

where $\Delta t = t_2 - t_1$.

Hazard rate the instantaneous rate,

$$h(t) = \lim_{\Delta t \to 0}\left[\frac{R(t) - R(t+\Delta t)}{\Delta t\, R(t)}\right] = -\frac{1}{R(t)}\frac{d}{dt}R(t) = \frac{f(t)}{R(t)}. \qquad (10)$$

Mean time to failure and mean time between failures

Mean time to failure is the mean time to first failure (O'Connor, 1985)

$$MTTF = \int_0^\infty R(t)dt.$$

Using integration by parts and property R (∞) = 0 give $MTTF = \int_0^\infty R(t)dt$. (11)

Mean time to failure should be used in the case of simple components which are not repaired when they fail but are replaced by good components.

Mean time between failures is the mean time between two successive component failures and should be used with repaired equipment or systems. Let T_i be the MTTF of i^{th} component of a system having n components, all of different ages. If each of which is replaced immediately on failure, then

$$\text{MTBF} = \frac{1}{\sum_{i=1}^{n} T_i}. \qquad (12)$$

It can be noted that for the useful life between t_1 and t_2, MTTF and MTBF are equal, and

(i) when the system is first operated with all new components, MTTF and MTBF will be identical.

The Normal Failure Law

There are many types of components whose failure behavior may be represented by the normal distribution. That is, if T is the life length (the time to failure) of item, its density is given by

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\frac{(t-\mu)^2}{\sigma^2}\right], \qquad (13)$$

where μ is the mean and σ the standard deviation.

Then we note that T must be greater than or equal to zero. As the shape of the normal density indicates, a normal failure law implies that the most of the item fail around the mean failure time, E (T) = μ (Figures- 1) and the number of failure decreases (symmetrically) as $|T - \mu|$ increases (Figure- 2).

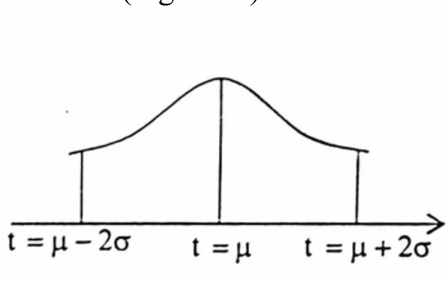

Figure (1) t satisfying $\{t \mid |T - \mu| < 2\sigma\}$

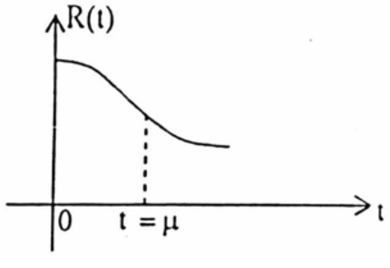

Figure (2) A general reliability curve

The reliability function of the normal failure law may be expressed in terms of the tabulated normal distribution function ϕ as follows:

$$R(t) = P(T > t) = 1 - P(T \le t)$$

$$= 1 - P\left(\frac{T-\mu}{\sigma} \le \frac{t-\mu}{\sigma}\right)$$

$$= 1 - \phi\left(\frac{t-\mu}{\sigma}\right). \qquad (14)$$

Example

Suppose that T, the time to failure of an item is normally distributed with E (T) equals 90 hours and standard deviation 5 hours. In order to achieve a reliability of 0.99, how many hours of operation may be considered?

Solution

E (T) = μ = 90, σ = 5. By using (14),

$$R(t) = P(T > t) = 0.99$$

$$1 - \phi\left(\frac{t-\mu}{\sigma}\right) = 0.99$$

$$\phi\left(\frac{t-90}{5}\right) = 0.01$$

$$\frac{t-90}{5} = -2.32$$

$$t = 78.4$$

Hence, we need 78.4 hours in order to achieve a reliability of 0.99.

The Exponential Failure Law

The exponential failure law is the one whose time to failure is described by the exponential distribution. It can be characterized in a number of ways, but the simplest way is to suppose that the failure rate is a constant. That is h(t) = α. (α is a constant)

An immediate consequence of this assumption is,

$$f(t) = h(t)e^{\int_0^t h(s)ds},$$

that the probability density function associated with the time to failure T, is given by

$$f(t) = \alpha e^{-\alpha t}, t > 0 .$$

Therefore, $R(t) = 1 - F(t) = 1 - \int_0^t f(s)ds = 1 - \left[e^{-\alpha s}\right]_0^t = e^{-\alpha t}$ and the exceptive time to failure, $E(T) = \frac{1}{\alpha}$ and $h(t) = \frac{f(t)}{R(t)} = \frac{\alpha e^{-\alpha t}}{e^{-\alpha t}} = \alpha$ (constant). (15)

Example

Suppose that the life length of an electronic device is exponentially distributed. It is known that the reliability of the device for a 100 hours period of operation is 0.90. How many hours of operation may be considered to achieve a reliability of 0.95?

Solution

t = 100, R(t) = 0.90. By using (15),

$$R(t) = e^{-\alpha t}$$

$$0.90 = e^{-100\alpha}$$

$$\alpha = \frac{\ln 0.90}{-100} = 0.0011\text{hours.}$$

$\alpha = 0.0011$, $R(t) = 0.95$,

$$0.95 = e^{-0.0011\,t}$$

$$t = \frac{\ln 0.95}{-0.0011} = 46.63 \text{ hours.}$$

The Weibull Failure Law

In 1951, Waloddi Weibull, a Swedish engineer introduced what he called "a statistical distribution function of wide applicability". He used, instead of a simple variable in the exponent, a polynomial term, (Wiliams, 1991)

$$F(t) = 1 - \exp\left[-\left(\frac{t}{c}\right)^m\right], t \geq 0$$

where the parameters c and m are positive. Differentiation gives

$$f(t) = \frac{m}{c}\left(\frac{t}{c}\right)^{m-1}\exp\left[-\left(\frac{t}{c}\right)^m\right], t \geq 0\,.$$

Therefore, the Weibull mean is

$$E(t) = \int_0^\infty tf(t)dt = \int_0^\infty m\left(\frac{t}{c}\right)^m \exp\left[-\left(\frac{t}{c}\right)^m\right]dt. \qquad (16)$$

Setting $y = \left(\frac{t}{c}\right)^m, dy = \frac{m}{c}\left(\frac{t}{c}\right)^{m-1} dt$

$$E(T) = \int_0^\infty mye^{-y}\frac{c}{m}y^{-1+\frac{1}{m}}dy = c\int_0^\infty y^{\frac{1}{m}}e^{-y}dy\,.$$

$$E(T) = c\Gamma\left(\frac{1}{m}+1\right),$$

where $\Gamma\left(\frac{1}{m}+1\right)$ is the gamma function of $\frac{1}{m}+1$. The reliability R(t) of the Weibull random variable is $R(t) = 1 - F(t)$

$$= \exp\left[-\left(\frac{t}{c}\right)^m\right],\ t \geq 0$$

$$h(t) = \frac{f(t)}{R(t)} = \left(\frac{m}{c}\right)\left(\frac{t}{c}\right)^{m-1}.$$

Its cumulative failure rate, H(t), is

$$H(t) = \int_0^t h(s)ds = \frac{m}{c}\int_0^t \left(\frac{s}{c}\right)^{m-1} ds = \frac{m}{c^m}\left[\frac{s^m}{m}\right]_0^t = \left(\frac{t}{c}\right)^m . \qquad (17)$$

Example

The manufacturer of a sonar system used by fishermen wants how many sonar systems will, on the average, fail in their first 9000h of normal use (9000h is approximately one year). Experience with the sonar system indicate that its failure rate can be modeled with a Weibull random variable having a shape parameter of m equals 0.3 and a time parameter of c equals 90.5×10^6 h.

Solution

$m = 0.3$, $c = 90.5 \times 10^6$ h.

The cumulative failure rate for Weibull model is $H(t) = \left(\frac{t}{c}\right)^m$.

$$H(9000h) = \left(\frac{9000}{90.5 \times 10^6}\right)^{0.3} = 0.06299$$

The manufacturer of the sonar system should expect that 6 out of each 100 units sold will, with use, fail in the first year.

The Exponential Failure Law and The Poisson Distribution

Let X be the number of disturbances occurring during a time interval of length t and suppose that X, $t \geq 0$, constitutes process. That is for any fixed t, the random variable X has a Poisson distribution with parameter α t. (Grimmett and Welsh, 1986)

Suppose that failure during [0, t] is caused iff at least one such disturbance occurs. Let T be the time to failure and assume to be a continuous random variable. Then

$$F(t) = 1 - p(T \leq t) = 1 - p(T > t).$$

Now , T > t iff no disturbance occurs during [0, t] and this happens iff X = 0.

Hence

$$F(t) = 1 - P(X = 0) = 1 - e^{\alpha t} . \qquad (18)$$

And this represents the distribution function of an exponential failure.

The above idea may be generalized as follows:

(a) Suppose that disturbances appear according to a Poisson process. Assume that whenever such a disturbance does not appear, there is a constant probability p that it will not cause failure. Therefore, if T represents the time to failure, we have

$F(t) = P(T \leq t) = 1 - P(T > t)$. This time, T > t iff no disturbance occurs, or one disturbance occurred and no failure resulted or two disturbance.

Hence

$$F(t) = 1 - [\, e^{-\alpha t} + (\alpha t)\, e^{-\alpha t} p + (\alpha t)^2\, e^{-\alpha t} p^2 + (\alpha t)^3\, e^{-\alpha t} p^3 + \ \ldots\ ]$$

$$= 1 - e^{-\alpha t}\ [\sum_{k=0}^{\infty} \frac{(\alpha t p)^k}{k!}]$$

$$= 1 - e^{-\alpha t}\ e^{\alpha t p}$$

$$= 1 - e^{-\alpha t\,(1-p)}.$$

Thus T has an exponential failure law parameter $\alpha(1 - p)$.

(b) Suppose that disturbances appear according to a Poisson process, and assuming that the failure occurs whenever "r" of more disturbances ($t \geq 1$) occur during an interval of length t.
If T is time to failure, Then $F(t) = 1 - p(T > t)$ and in this case $T > t$ iff

(r – 1) fewer disturbances occur.

Therefore

$$F(t) = 1 - \sum_{k=0}^{r-1} \frac{(\alpha t)^k e^{-\alpha t}}{k!}$$

$$= \int_0^t \frac{\alpha}{(\gamma - 1)!} (\alpha s)^{n-1} e^{n-1}\, ds\ , \qquad (19)$$

which is the distribution of a Gamma distribution. Thus the above "cause of failure" leads to the conclusion that time to failure follow a Gamma failure law.

Reliability of Systems

Structure functions

Consider a system consisting of n components, and suppose that each component is either functioning or has failed. To indicate whether or not the i^{th} component is functioning, we define the indicator variable x_i by (Tobias, et al., 1986)

$$x_i = \begin{cases} 1, & \text{if the } i^{th} \text{ component is functioning} \\ 0, & \text{if the } i^{th} \text{ component has failed.} \end{cases}$$

The vector $\underline{x} = (x_1, x_2, \ldots, x_n)$ is called the state vector.

Suppose that whether or not the system as a whole is functioning is completely determined by the state vector $\underline{x}$.

It is supposed that there exists the structure function $\phi(x)$ such that

$$\phi(x) = \begin{cases} 1, & \text{if the system is functioning when the state vector is } \underline{x} \\ 0, & \text{if the system has failed when the state vector is } \underline{x}\ . \end{cases}$$

A system will consist of a finite number of components connector in series or in parallel or in combination of series and parallel. The concept of system reliability is a measure of how well the system performs or meets its design objects.

The series and parallel systems are both special cases of a k-out-of-n system. Such a system functions if and only if at least k out of the n components are functioning. The structure function is given by

$$\phi(x) = \begin{cases} 1, \text{ if } \sum_{i=1}^{n} x_i \geq k \\ 0, \text{ if } \sum_{i=1}^{n} x_i < k. \end{cases}$$

Series System

A series system is a system (Figure- 3) in which all components are so connected that the entire system will fail if any one of its components fails. The series system is the most commonly encountered model and the simplest one.

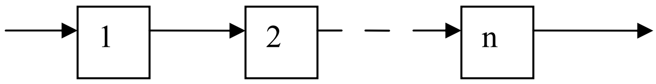

Figure (3) n components are hooked up in series

If n components, functioning independently, are connecter in series, and if the i[th] component has reliability $R_i(t)$, then the reliability of the entire system, $R_s(t)$, is given by

$$R_s(t) = \prod_{i=1}^{n} R_i(t).$$

Then $$\ln[R_s(t)] = \sum_{i=1}^{n} \ln[R_i(t)]. \qquad (20)$$

Differentiating both sides of the (20) with respect to time

$$\frac{R_s'(t)}{R_s(t)} = \sum_{i=1}^{n} \frac{R_i'(t)}{R_i(t)},$$

and $$h(t) = \sum_{i=1}^{n} (-h_i(t))(-h_i(t)) \text{ and hence } h_s(t) = \sum_{i=1}^{n} h_i(t).$$

The n components have the constant failure α_i corresponding to exponential random variable

$$h_s(t) = \sum_{i=1}^{n} \alpha_i = \alpha_s$$

and the system reliability $R_s(t) = \prod_{i=1}^{n} R_i(t)$ becomes $R_s(t) = e^{-\alpha_s t}$.

Therefore the mean time between failures for the series system is

$$\text{MTBF} = \int_0^\infty R_s(t)dt = \int_0^\infty e^{-\alpha_s t}dt = \frac{1}{\alpha_s}. \tag{21}$$

If $\alpha_1 = \alpha_2 = \ldots = \alpha_n = \alpha$, then $\text{MTBF} = \dfrac{1}{n\,\alpha}$.

Example

Consider an electronic circuit consisting of 4 silicon transistors, 10 silicon diodes, 20 composition resistors, and 10 ceramic capacitors in continuous series operation.

Suppose that under certain stress conditions, each of these items has the following constant failure rates:

silicon transistors : 0.00001;

silicon diodes : 0.000002;

composition resistors : 0.000001;

ceramic capacitors : 0.000002.

Find the exceptive time to failure. And also find the reliability function for the system for a 10 hours period of operation.

Solution

silicon transistors :	0.00001	x	4	=	0.00004
silicon diodes :	0.000002	x	10	=	0.00002
composition resistors :	0.000001	x	20	=	0.00002
ceramic capacitors :	0.000002	x	10	=	0.00002
					0.00010

$\alpha = 0.00010$,

Exceptive time to failure $\quad E(T) = \dfrac{1}{\alpha} = 10000$ hours.

Reliability function for the system $\quad R(t) = e^{-\alpha t} = e^{-0.0001\,t}$

For 10 hours period of operation $\quad R(10) = 0.999.$

Parallel system

A parallel system is a system that is not considered to have failed unless all components have failed. The reliability block diagram can be given in figure (4).

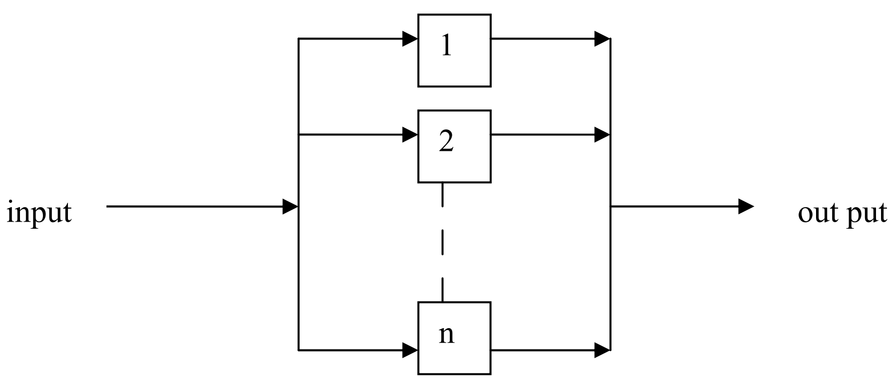

Figure (4) Parallel System

If n components functioning independently are operating in parallel, and if the i^{th} component has reliability $R_i(t)$, then the reliability if the system, $R_p(t)$, is given by

$$R_p(t) = 1 - [\,(1-R_1(t))(1-R_2(t))\ldots(1-R_n(t))]$$

$$= 1 - \prod_{i=1}^{n} (1 - R_1(t)). \qquad (22)$$

In particular, if we consider 2 components parallel, system assuming the subsystem 1 and 2 function independently. Then

$$R_p(t) = 1 - (1-R_1(t))(1-R_2(t))$$

$$= R_1(t) + R_2(t) - R_1(t)\, R_2(t).$$

If the failure distribution of the system is

$$F_p = F_1(t)\, F_2(t)$$

$$= (1-R_1(t))(1-R_2(t)).$$

If $R_1(t) = e^{-\alpha_1 t}$, then $R_p(t) = e^{-\alpha_1 t} + e^{-\alpha_2 t} - e^{-(\alpha_1+\alpha_2)t}$.

And mean time failure is

$$\text{MTTF} = \int_0^\infty R_p(t)dt$$

$$\frac{1}{\alpha_p} = \int_0^\infty (e^{-\alpha_1 t} + e^{-\alpha_2 t} - e^{-(\alpha_1+\alpha_2)t})\, dt$$

$$\frac{1}{\alpha_p} = \frac{1}{\alpha_1} + \frac{1}{\alpha_2} - \frac{1}{\alpha_1 + \alpha_2}$$

where α_p is the failure rate for the parallel system.

Also $$f_p(t) = \frac{d}{dt} F_p(t)$$

$$= f_1(t)\, F_2(t) + F_2(t)\, F_1(t)$$

$$= (\alpha_1 e^{-\alpha_1 t})(1-e^{-\alpha_2 t}) + (\alpha_2 e^{-\alpha_2 t})(1 - e^{-\alpha_1 t})$$

$$f_p(t) = \alpha_1 e^{-\alpha_1 t} + \alpha_2 e^{-\alpha_2 t} - (\alpha_1 + \alpha_2)\, e^{-(\alpha_1+\alpha_2)t}.$$

And we have $$h_p(t) = \frac{f_p(t)}{R_p(t)}$$

$$= \frac{\alpha_1 e^{-\alpha_1 t} + \alpha_2 e^{-\alpha_2 t} - (\alpha_1 + \alpha_2) e^{-(\alpha_1+\alpha_2)t}}{e^{-\alpha_1 t} + e^{-\alpha_2 t} - e^{-(\alpha_1+\alpha_2)t}}.$$

Example

Suppose the electrical power distribution to a city is obtained from two independent transmission lines with failure of the transmission lines are 0.02 and 0.015, respectively. The lines are connected in parallel at the city so that if one line is interrupted the other will continue to supply the city with power. The failure (interruption) rates for the lines are assumed to be exponential. Find the probability function, reliability function, hazard function and E(T) for the system for one year operation period.

Solution

Transmission line A : $\alpha_1 = 0.02$ failures per year.

Transmission line B : $\alpha_2 = 0.015$ failures per year.

The parallel system of transmission lines has the reliability

$$R(t) = e^{-\alpha_1 t} + e^{-\alpha_2 t} - e^{-(\alpha_1+\alpha_2)t}$$

$$= e^{-0.02t} + e^{-0.015t} - e^{-0.035t}$$

$$f(t) = F'(t) = [\,1 - R(t)\,]'$$

$$= 0.02\, e^{-0.02t} + 0.015\, e^{-0.015t} - 0.035\, e^{-0.035t}.$$

For one year,

$$f(t) = 0.02\, e^{-0.02\,(1)} + 0.015\, e^{-0.015\,(1)} - 0.035\, e^{-0.035\,(1)}$$

$$= 0.000584462.$$

$$R(t) = e^{-0.02\,(1)} + e^{-0.015\,(1)} - e^{-0.035\,(1)}$$

$$= 0.999705196.$$

$$h(t) = \frac{f(t)}{R(t)} = \frac{0.000584462}{0.999705196} = 0.000584175\,.$$

Mean time to failure of the system

$$E(T) = \frac{1}{\alpha_1} + \frac{1}{\alpha_2} - \frac{1}{\alpha_1 + \alpha_2}$$

$$= \frac{1}{0.02} + \frac{1}{0.015} - \frac{1}{0.035}$$

$$= 88.1 \text{ years.}$$

Conclusion

Reliability theory has been developed for controlling the quality of a machine, making designs and many others. We hope that the information and observations of this paper will be useful for finding the probability of failures in the given time interval. It will support to study the mean time to failures on the system of components and the mean time between failures for two successive components.

Acknowledgements

We would like to express our sincere thanks to Dr. San Linn, Acting Rector, Hinthada University, and Dr. Aung Win, Pro-Rector, Hinthada University, for their permission to carry out this research. We are greatly indebted to Daw Aye Aye, Professor and Head, Department of Mathematics, Hinthada University, for her kind permission and encouragement to carry out this work. We are also grateful to Dr. Moe Thu, Professor, Department of Mathematics, Hinthada University, for her valuable suggestions.

References

Grimmett, G. and Welsh, D. (1986) *Probability an Introduction*, Oxford University Press, New York.

Meyer, P. L. (1966) *Introductory Probability and Statistical Applocayions*, Addison Wesley Publishing Company. Inc., New York.

O'Connor, Patrick, D. T. (1985) *Practical Reliability Engineering*, 2nd edition, Chichester: John Wiley and Sons Ltd., London.

Tobias, Paul A., and David, C., Trinidade (1986) *Applied Reliability*, Van Nostrand Company, New York.

Wiliams, Richard H. (1991) *Electrical Engineering Probability*. St. Paul: West Publishing Company, London.