Averaged Description of Waves in the Korteweg-De Vries-Burger Equation

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Abstract

A perturbed Korteweg-de Vries equation is considered. We appeal to the multi-scale method of Luke (1966) and Ablowtiz & Benny (1970) in which the solution u (x ,t) is assumed to be a function of a fast variable θ and the slow time and space variables T, X and is periodic in θ and can be expressed as a formal power series in powers of ε , a small positive parameter. We obtain a nonlinear nonhomogeneous system of first order partial differential equations for the parameters of the wave train, such as the amplitude, the average depth, and the wave number. Although the perturbation term can in general be left arbitrary, we deal specifically with the frictional term representing KdV-Burgers damping. The initial condition is a step discontinuity which evolves into a disturbance resembling an undular bore. Using the modulation theory we have developed, we find that at the region just behind the leading-front, the amplitude, the phase speed and the average depth of the waves increases; but at the region just ahead of the trailing-edge, the amplitude of the waves decays as the reciprocal of the slow time.

Key words: consistency condition, frequency, period, phase, wave number

Introduction

The problem of modulated nonlinear periodic waves is described by the perturbed Korteweg – de Vries equation,

$$u_t + uu_x + u_{xxx} + \varepsilon V(u) = 0, \qquad (1)$$

where V (u) is an arbitrary functional of u (x , t) , ε is a small positive number measuring the strength of the frictional term V (u) and subscripts denote partial differentiations. Equation of the form (1) occurs in many circumstances.

When $V(u) = -\gamma\, u_{xx}$, $\gamma > 0$ so that (1) is the Korteweg-de Vries -Burgers equation (Karpman, 1975). To study undular bores which are defined here in general sense as the solution of the perturbed Korteweg-de Vries equation (1), with the initial condition being the step discontinuity

$$u(x,0) = \begin{cases} h_0, & \text{when } x < 0 \\ 0, & \text{when } x > 0 \end{cases} \qquad (2)$$

where h_0 is a positive constant and for the case of the specific frictional term $V(u) = -\gamma\, u_{xx}$.

Fast and Slow Variables

Following Luke (1966), Ablowitz & Benny (1970), Whitham (1974), we introduced the fast variable θ and the slow space and time variables X and T by

$$\theta = \varepsilon^{-1}\,\Theta(X, T), \qquad X = \varepsilon x, \qquad T = \varepsilon t, \qquad (3)$$

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so that $u = u(\theta, X, T)$. This is analogous to the procedure used by Kuzmak (1959) for nonlinear oscillations described by ordinary differential equations. The local wave number $\kappa(X, T)$, frequency $\omega(X, T)$ and phase speed U are defined by

$$\kappa = \theta_x = \Theta_X \quad , \quad \omega = -\theta_t = -\Theta_T \quad , \quad c = U = \frac{\omega}{\kappa} \, , \tag{4}$$

from which we get the consistency condition , (which is known as the conservation of waves)

$$\kappa_T + \omega_X = 0. \tag{5}$$

Asymptotic Expansion

We shall use the perturbation scheme developed for slowly varying solitary waves by Grimshaw (1970), Johnson (1973) and Ko & Kuehl (1978). We seek an asymptotic solution $u(\theta, X, T)$ of the form

$$u(\theta, X, T) = u_0(\theta,X,T) + \varepsilon u_1(\theta,X,T) + \varepsilon^2 u_2(\theta,X,T) + \cdots \, . \tag{6}$$

Then, we have $V(u) = V(u_0) + V_u(u_0)\{\varepsilon u_1 + \varepsilon^2 u_2 + \cdots\} + \cdots$. By using asymptotic expansion (6), relations (4) and (5) , in the perturbed KdV equation (1) we get

$$-\omega u_\theta + \varepsilon u_T + u(\kappa u_\theta + \varepsilon u_X) + (\kappa^3 u_{\theta\theta\theta} + 3\varepsilon\kappa^2 u_{\theta\theta X} + 3\varepsilon\kappa\kappa_X u_{\theta\theta} + 3\varepsilon^2 \kappa u_{\theta XX} + 3\varepsilon^2 \kappa_X u_{\theta X} + \varepsilon^2 \kappa_{XX} u_\theta + \varepsilon^3 u_{XXX}) + \varepsilon\,(V(u_0) + V_u(u_0)\{\varepsilon u_1 + \varepsilon^2 u_2 + \cdots\} + \cdots . = 0 \tag{7}$$

Leading – Order Expansion

From the expressions (6) , (7) , and for the leading – order , we have third order nonlinear differential equation for u_0 ,

$$-\omega u_{0\theta} + \kappa u_0 u_{0\theta} + \kappa^3 u_{0\theta\theta\theta} = 0, \tag{8}$$

which can be treated as an ordinary differential equation with independent variable θ.

Periodic Solution

From equation (8), we can rewrite

$$-U u_{0\theta} + u_0 u_{0\theta} + \kappa^2 u_{0\theta\theta\theta} = 0, \tag{9}$$

and following Korteweg – de Vries (1895) , we assume the solution of the form

$$u_0 = a\,(b + cn^2(\beta\theta)) + d\,, \tag{10}$$

where the modular cosine function cn (y, s^2) is a Jacobi elliptic function of modulus s. Equations (9) and (10) give the relations

$$a = 12\,\kappa^2\beta^2 s^2 = \frac{U - d + 4\kappa^2\beta^2}{b + (2/3)} \, , \tag{11}$$

and then the periodic solution (10) reduces to the form

$$u_0 = \frac{a}{s^2}dn^2(\beta\theta) + U - \frac{a}{3s^2}(2-s^2) \quad , \tag{12}$$

where the modular amplitude function dn (y, s^2) is a Jacobi elliptic function of modulus s, (see Abramowitz & Stegun, 1965) .The periodic of u_0 is 2P , $K(s^2)$ is the complete elliptic integral of the first kind and $a/2$ determines the amplitude of the oscillations:

$$a = u_{0\max} - u_{0\min} \; . \tag{13}$$

In particular, when the phase $\theta = \kappa X - \omega T$, by using (11),the periodic solution u_0 becomes

$$u_0 = \frac{a}{s^2}dn^2\left[\left(\frac{a}{12s^2}\right)^{1/2}(X-UT), s^2\right] + U - \frac{a}{3s^2}(2-s^2). \tag{14}$$

By integrating (9) with respect to the fast variable θ once and twice respectively, we get

$$-Uu_0 + \frac{1}{2}u^2{}_0 + \kappa^2 u_{0\theta\theta} - B = 0 , \tag{15}$$

$$-Uu^2{}_0 + \frac{1}{3}u^3{}_0 + \kappa^2(u_{0\theta})^2 - 2Bu_0 - 2A = 0 , \tag{16}$$

where the constants of integration A(X, T) and B(X , T) are given by the relations

$$A = \frac{1}{2}\left\{-\frac{2}{3}(ab+d)^3 + U(ab+d)^2 - 4a\beta^2\kappa^2(ab+d)(1-s^2)\right\} , \tag{17}$$

$$B = \frac{1}{2}(ab+d)^2 - U(ab+d) + 2a\beta^2\kappa^2(1-s^2) \; . \tag{18}$$

Polynomial and Some Relations

Again, we write equation (16) in the form

$$\kappa^2 u^2{}_{0\theta} = 2A + 2Bu_0 + Uu^2{}_0 - \frac{u^3{}_0}{3} \; . \tag{19}$$

Let p, q, r ; (r < q < p), be the real zeros of the polynomial on the right hand side of (19). Then we have the relation $q < u_0 < p$ and the periodic solution u_0 takes the form

$$u_0 = q + (p-q)\, cn^2\left(\sqrt{\frac{p-r}{12\kappa^2}}\theta\right) , \tag{20}$$

with the relations

$$A = \frac{1}{6}pqr \; , \; B = -\frac{1}{6}(pq+qr+rp) \quad , \quad U = \frac{1}{3}(p+q+r), \tag{21}$$

$$a\,b + d = q\,, \quad a = p - q\,, \quad a = 12\,\kappa^2\beta^2 s^2\,, \quad r = U - \frac{a}{3s^2}\,(2 - s^2\,), \tag{22}$$

$$\beta = \sqrt{\frac{a}{12 s^2\kappa^2}} = \sqrt{\frac{p-q}{12 s^2\kappa^2}}\,, \quad s^2 = \frac{a}{12\beta^2\kappa^2} = \frac{p-q}{12\beta^2\kappa^2}\,, \tag{23}$$

$$d = \frac{a}{s^2}\frac{E(s^2)}{K(s^2)} + r = \bar{u}_0 = \frac{1}{2P}\int_{-P}^{P} u_0\, d\theta, \quad \int_{-P}^{P}\frac{1}{2}u^2{}_0\ d\theta = 2P\,(\,U\,d + B\,), \tag{24}$$

$$\int_{-P}^{P}[\frac{1}{3}u^3{}_0 - \frac{3}{2}(u_{0\theta})^2]\ d\theta = 2P\,\{\,U\,(\,U\,d + B\,) - A\,\}, \tag{25}$$

where $E(s^2)$ is the complete elliptic integral of the second kind.

First Order Expansion

From the expanded equation (7), if we equate coefficients of ε to zero, we get the third order linear differential equation for:

$$-\,\omega\, u_{1\theta} + \kappa(u_0 u_1)_\theta + \kappa^3\, u_{1\theta\theta\theta} = f_1\,, \tag{26}$$

where the expression f_1 is given by

$$f_1 = -\,u_{0T} - u_0 u_{0X} - 3\,\kappa^2 u_{0\theta\theta X} - 3\,\kappa\,\kappa_X u_{0\theta\theta} - V\,(\,u_0\,). \tag{27}$$

Periodicity Conditions

In order for u_1 to be periodic in θ with periodic 2P, we should have the conditions

$$\int_{-P}^{P} f_1\, d\theta = 0\,, \quad \int_{-P}^{P} u_0 f_1\, d\theta = 0, \tag{28}$$

and by using f_1 from (27), the first condition in (28) simplifies to the condition

$$\frac{\partial}{\partial T}\left(\int_{-P}^{P} u_0 d\theta\right) + \frac{\partial}{\partial X}\left(\int_{-P}^{P}\frac{1}{2}u^2{}_0 d\theta\right) = -\int_{-P}^{P} V(u_0)\ d\theta, \tag{29}$$

and by using from (27), the second condition in (28) simplifies to the condition

$$\frac{\partial}{\partial T}\left(\int_{-P}^{P}\frac{1}{2}u^2{}_0 d\theta\right) + \frac{\partial}{\partial X}\left(\int_{-P}^{P}\left[\frac{1}{3}u^3{}_0 - \frac{3}{2}\kappa^2(u_{0\theta})^2\right]d\theta\right) = -\int_{-P}^{P} u_0 V(u_0)\ d\theta. \tag{30}$$

By using the conditions (24), the periodicity condition (29) becomes

$$\frac{\partial}{\partial T}(d) + \frac{\partial}{\partial X}(Ud+B) = -\frac{1}{2P}\int_{-P}^{P} V(u_0)\ d\theta, \qquad (31)$$

and by using the conditions (24) (25) , the periodicity condition (30) becomes

$$\frac{\partial}{\partial T}(Ud+B) + \frac{\partial}{\partial X}(\{U(Ud+B)-A\}) = -\frac{1}{2P}\int_{-P}^{P} u_0 V(u_0)\ d\theta. \qquad (32)$$

Function W and Related Relations

In order to put the consistency condition (5) and the periodicity conditions (31), (32) in a more symmetric form, we introduced the function W(A, B, U) by the relation

$$W = \frac{\kappa}{2P}\oint u_{0\theta}du_0 = \frac{1}{2P}\oint\sqrt{2A+2BU_0+Uu^2{}_0-u^3{}_0/3}\ du_0. \qquad (33)$$

Here, the symbol $\oint$ denotes integration over a complete cycle of oscillation. We also noticed that in the definition (33), the function u_0 plays the role of a dummy variable . Also we can easily obtain the relations

$$\kappa = W^{-1}{}_A, \quad d = W^{-1}{}_A W_B, \quad U\,d+B = W^{-1}{}_A W_U. \qquad (34)$$

Since p, q, r are the zeros of $\sqrt{2A+2Bu_0+Uu^2{}_0-u^3{}_0/3}$, we can express W(p ,q ,r) in the form

$$W = \frac{1}{2P}\frac{1}{\sqrt{3}}\int_q^p\sqrt{(p-u_0)(u_0-q)(u_0-r)}\ du_0. \qquad (35)$$

Following Byrd & Friedman (1971), we also can express the functions W, W_A and W_B in terms of the two complete elliptic integrals of the first and second kind $K(s^2)$, $E(s^2)$ by the relations

$$W = \frac{1}{2P}\frac{8}{15\sqrt{3}}\left(\frac{a}{s^2}\right)^{5/2}\{(1-s^2+s^4)E(s^2)-(1-s^2)(1-s^2/2)K(s^2)\}, \qquad (36)$$

$$W_A = \frac{1}{P}\left(\frac{12s^2}{a}\right)^{1/2}K(s^2),\quad W_B = \frac{1}{P}\left(\frac{12s^2}{a}\right)^{1/2}K(s^2)\left\{r+\frac{a}{s^2}\frac{E(s^2)}{K(s^2)}\right\}. \qquad (37)$$

Modulation Equations for the Perturbed KdV Equation

By using the relations (34), the consistency condition (5), the periodicity conditions (31) and (32) reduce to the modulation equations in the more symmetric form:

$$\frac{\partial W_A}{\partial T} + U\frac{\partial W_A}{\partial X} - W_A\frac{\partial U}{\partial X} = 0 \,, \tag{38}$$

$$\frac{\partial W_B}{\partial T} + U\frac{\partial W_B}{\partial X} + W_A\frac{\partial B}{\partial X} = -\frac{W_A}{2P}\int_{-P}^{P} V(u_0)\, d\theta, \tag{39}$$

$$\frac{\partial W_U}{\partial T} + U\frac{\partial W_U}{\partial X} - W_A\frac{\partial A}{\partial X} = -\frac{W_A}{2P}\int_{-P}^{P} u_0 V(u_0)\, d\theta. \tag{40}$$

These three modulation equations (38) ,(39) and (40) were indeed obtained by Whitham [13] for the case when $V(u) = 0$. To simplify equations (38) ,(39) and (40) , we noticed that $W = W(p, q, r)$ and we evaluate W_A, W_B, W_U in terms of W_p, W_q, W_r to get the relations

$$W_A = -6\left\{\frac{W_p}{(p-q)(r-p)} + \frac{W_q}{(p-q)(q-r)} + \frac{W_r}{(q-r)(r-p)}\right\}, \tag{41}$$

$$W_B = -6\left\{\frac{pW_p}{(p-q)(r-p)} + \frac{qW_q}{(p-q)(q-r)} + \frac{rW_r}{(q-r)(r-p)}\right\}, \tag{42}$$

$$W_U = -3\left\{\frac{p^2W_p}{(p-q)(r-p)} + \frac{q^2W_q}{(p-q)(q-r)} + \frac{r^2W_r}{(q-r)(r-p)}\right\}. \tag{43}$$

By using the relations (41) , (42) and (43) in equations (38) , and after simplifying and arranging terms (nontrival) , we get the equation

$$(p+q)_T + \frac{1}{3}\left\{(p+q+r) - \frac{p(W_q - W_r) + q(W_r - W_p) + r(W_p - W_q)}{(W_p - W_q)}\right\}(p+q)_X$$

$$= \frac{(p-q)}{6(W_p - W_q)}\left\{\frac{rW_A}{P}\int_{-P}^{P} V(u_0)d\theta - \frac{W_A}{P}\int_{-P}^{P} u_0 V(u_0)d\theta\right\}, \tag{44}$$

the equation

$$(q+r)_T + \frac{1}{3}\left\{(p+q+r) - \frac{p(W_q - W_r) + q(W_r - W_p) + r(W_p - W_q)}{(W_q - W_r)}\right\}(q+r)_X$$

$$= \frac{(q-r)}{6(W_q - W_r)}\left\{\frac{pW_A}{P}\int_{-P}^{P} V(u_0)d\theta - \frac{W_A}{P}\int_{-P}^{P} u_0 V(u_0)d\theta\right\}, \qquad (45)$$

and the equation

$$(r+p)_T + \frac{1}{3}\left\{(p+q+r) - \frac{p(W_q - W_r) + q(W_r - W_p) + r(W_p - W_q)}{(W_r - W_p)}\right\}(r+p)_X$$

$$= \frac{(r-p)}{6(W_r - W_p)}\left\{\frac{qW_A}{P}\int_{-P}^{P} V(u_0)d\theta - \frac{W_A}{P}\int_{-P}^{P} u_0 V(u_0)d\theta\right\}. \qquad (46)$$

Modulation Equations in Characteristic Form

Following Whitham (1974), we introduced the variables r_1 , r_2 and r_3 through the relations

$$r_1 = q + r, \qquad r_2 = r + p, \qquad r_3 = p + q, \qquad (47)$$

so that the equations (44), (45), (46), can be put into a nonlinear nonhomogeneous system of first – order partial differential equations

$$\frac{\partial r_1}{\partial T} + Q_1(r_1, r_2, r_3)\frac{\partial r_1}{\partial X} = M_1(r_1, r_2, r_3), \qquad (48)$$

$$\frac{\partial r_2}{\partial T} + Q_2(r_1, r_2, r_3)\frac{\partial r_2}{\partial X} = M_2(r_1, r_2, r_3), \qquad (49)$$

$$\frac{\partial r_3}{\partial T} + Q_3(r_1, r_2, r_3)\frac{\partial r_3}{\partial X} = M_3(r_1, r_2, r_3), \qquad (50)$$

where the characteristic velocities Q_1, Q_2, Q_3, are given by the expressions

$$Q_1 = \frac{1}{6}(r_1 + r_2 + r_3) - \frac{a}{3}\frac{K(s^2)}{\{K(s^2) - E(s^2)\}}, \qquad (51)$$

$$Q_2 = \frac{1}{6}(r_1 + r_2 + r_3) - \frac{a}{3}\frac{(1-s^2)K(s^2)}{\{E(s^2) - (1-s^2)K(s^2)\}}, \qquad (52)$$

$$Q_3 = \frac{1}{6}(r_1 + r_2 + r_3) + \frac{a}{3}\frac{(1-s^2)K(s^2)}{s^2 E(s^2)}, \qquad (53)$$

and the nonhomogeneous terms M_1, M_2, M_3 are given by the expressions

$$M_1 = -\frac{1}{P}\int_{-P}^{P} V(u_0)du_0 + \frac{s^2K(s^2)}{a\{K(s^2)-E(s^2)\}}\,\frac{1}{P}\int_{-P}^{P}(u_0-d)V(u_0)du_0\,, \qquad (54)$$

$$M_2 = -\frac{1}{P}\int_{-P}^{P} V(u_0)du_0 - \frac{s^2K(s^2)}{a\{E(s^2)-(1-s^2)K(s^2)\}}\,\frac{1}{P}\int_{-P}^{P}(u_0-d)V(u_0)du_0\,, \qquad (55)$$

$$M_3 = -\frac{1}{P}\int_{-P}^{P} V(u_0)du_0 - \frac{s^2K(s^2)}{aE(s^2)}\,\frac{1}{P}\int_{-P}^{P}(u_0-d)V(u_0)du_0\,. \qquad (56)$$

Modulation Equations for the KdV-BURGERS Equation

For the case $V(u) = -\gamma\, u_{XX}$, $(\gamma > 0)$, the case of Korteweg-de Vries-Burgers equation

$$u_t + uu_x + u_{xxx} - \varepsilon\gamma\, u_{xx} = 0, \qquad (57)$$

we simplify the nonhomogeneous terms M_1, M_2, M_3 , given by (54), (55), (56) to get the expressions

$$M_1 = \frac{4\gamma a^2}{45s^4}\,\frac{\{(1-s^2+s^4)E(s^2)-(1-s^2)(1-s^2/2)K(s^2)\}}{\{K(s^2)-E(s^2)\}}\,, \qquad (58)$$

$$M_2 = \frac{-4\gamma a^2}{45s^4}\,\frac{\{(1-s^2+s^4)E(s^2)-(1-s^2)(1-s^2/2)K(s^2)\}}{\{E(s^2)-(1-s^2)K(s^2)\}}\,, \qquad (59)$$

$$M_3 = \frac{-4\gamma a^2}{45s^4}\,\frac{\{(1-s^2+s^4)E(s^2)-(1-s^2)(1-s^2/2)K(s^2)\}}{E(s^2)}\,. \qquad (60)$$

Initial Conditions for the Modulation Equations

We introduced the initial step discontinuity, (initial condition for u (x .t) at t = 0):

$$u(x,0) = \begin{cases} h_0, & \text{when } x < 0 \\ 0, & \text{when } x > 0 \end{cases} \qquad (61)$$

where h_0 is a positive constant . Following Gurevich & Pitaevskii (1973), we can deduce the initial conditions for the modulation equations (48), (49), (50) at
T = 0:

$$\left.\begin{aligned} r_1 = r_2 = 0\,,\quad r_3 = 2h_0\,, \quad &\text{for}\quad X < 0\,, \\ r_1 = 0\,,\quad r_2 = r_3 = 2h_0\,, \quad &\text{for}\quad X > 0. \end{aligned}\right\} \qquad (62)$$

For the case V (u) = $-\gamma\ u_{xx}$, with the initial condition being the step discontinuity (61) , we shall use the modulation theory developed , with the initial conditions (62) , to study the behavior of waves at those regions just behind the leading front and ahead of the trailing edge .

Solution for the Leading Front

At the leading front where $s^2 \to 1$, $r_2 = r_3$, q = r, by using the initial condition (61) in

$$d = \frac{1}{6}(r_1 + r_2 + r_3) - \frac{1}{3}(r_3 - r_1)\left(2 - s^2 - 3\frac{E(s^2)}{K(s^2)}\right), \tag{63}$$

the result which is obtained from (21), (22) , (23) , we get

$$r_1 = 0, \quad r_2 = r_3 = P(T), \qquad a = P(T), \tag{64}$$

where we assumed that the solution in this region is a function of T alone . Then we have from (51), (52) , (53) , (58) , (59) , (60) , and by noticing $E(s^2) \to 1$, $K(s^2) \to +\infty$, $(1-s^2)K(s^2) \to 0$, $K(s^2)/\{K(s^2) - E(s^2)\} \to 1$ as $s^2 \to 1$, we have

$$Q_1 \sim 0, \qquad Q_2 \sim \frac{1}{3}P(T), \qquad Q_3 \sim \frac{1}{3}P(T),$$

and

$$M_1 \sim 0, \qquad M_2 \sim -\frac{4\gamma}{45}P^2(T), \qquad M_3 \sim -\frac{4\gamma}{45}P^2(T),$$

and the system (48) , (49) , (50) , reduced to a single equation

$$\frac{\partial P}{\partial T} + \frac{1}{3}P\frac{\partial P}{\partial X} = -\frac{4\gamma}{45}P^2(T),$$

and hence we obtained that $p = (c + 4T\gamma/45)^{-1}$. But, when $\gamma = 0$, (T = 0) , and when $s^2 \to 1$, we had seen that $r_2 = 2h_0$, (see Gurevich & Pitaevskii, 1973) ,so that $c = 1/2h_0$, with the result $P = 2h_0(1 + 8h_0T\gamma/45)^{-1}$ and the relations

$$Q_1 \sim 0, \qquad Q_2 \sim \frac{2h_0}{3}\left(1 + \frac{8h_0T\gamma}{45}\right)^{-1}, \qquad Q_3 \sim \frac{2h_0}{3}\left(1 + \frac{8h_0T\gamma}{45}\right)^{-1},$$

and the relations

$$M_1 \sim 0, \qquad M_2 \sim -\frac{16h^2{}_0\gamma}{45}\left(1 + \frac{8h_0T\gamma}{45}\right)^{-2}, \qquad M_3 \sim -\frac{16h^2{}_0\gamma}{45}\left(1 + \frac{8h_0T\gamma}{45}\right)^{-2}.$$

Then, at the leading front , we found that

$$r_1 = 0 , \quad r_2 = r_3 = 2h_0\left(1+\frac{8h_0T\gamma}{45}\right)^{-1} , \tag{65}$$

if the solution in this region is a function of T alone . The characteristic at the leading front is given by the differential equation $dX/dT = Q_2 = \frac{2h_0}{3}\left(1+\frac{8h_0T\gamma}{45}\right)^{-1}$ or by the curve

$$X = \frac{15}{4\gamma}\ln\left(1+\frac{8h_0T\gamma}{45}\right). \tag{66}$$

The characteristic (66) becomes $X \to (2h_0/3)T$, the result for the case when $V(u) = 0$, but as $T \to +\infty$, $X \to (15/4\gamma)\ \ln\left(1+\frac{8h_0T\gamma}{45}\right)$.

Solution for the Trailing Edge

At the trailing edge where $s^2 \to 0$, $r_1 = r_2$, $a \to 0$, p = q, by using the initial condition (61) in (63), we get

$$r_3 = 2\ h_0 , \quad r_1 = r_2 = h_0 + r(T) , \tag{67}$$

where we assumed that the solution in this region is a function of T alone .

Then we have from (51), (52), (53), (58), (59), (60), we get

$Q_1 \sim \gamma$, $Q_2 \sim \gamma$, $Q_3 \sim h_0$, and $M_1 \sim 0$, $M_2 \sim 0$, $M_3 \sim 0$,

and the system (48), (49), (50), reduced to a single equation

$$\frac{\partial\gamma}{\partial T} + \gamma\frac{\partial\gamma}{\partial X} = 0,$$

and hence γ = constant . But , when $\gamma = 0$, (T = 0) and when $s^2 \to 0$, we had seen that $r_1 = r_2 = 0$, (see Gurevich & Pitaevskii, 1973) , so that γ = constant = $-h_0$ and we are left with the relations

$Q_1 \sim - h_0$, $Q_2 \sim - h_0$, $Q_3 \sim h_0$, and $M_1 \sim 0$, $M_2 \sim 0$, $M_3 \sim 0$.

Then , at the trailing edge , we found that

$$r_1 = r_2 = 0 , \quad r_3 = 2\ h_0 , \tag{68}$$

if the solution in this region is a function of T alone . The characteristic at the trailing edge is given by the differential equation $dX/dT = Q_2 = - h_0$ or by the straight line

$$X = - h_0 T. \tag{69}$$

The characteristic (69) becomes $X \to - h_0 T$, the result for the case when $V(u) = 0$, but as $T \to +\infty$, $X \to -\infty$.

Conclusion

By using the asymptotic expansion, the consistency condition and the periodicity conditions, the modulation equations are obtained in the more symmetric form and characteristic form.

Using the modulation theory, we find that the behavior of waves is the curve $X = \frac{15}{4\gamma}\ln\left(1+\frac{8h_0T\gamma}{45}\right)$ at the leading-front and is the strongest line $X = - h_0 T$ at the trailing edge.

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