# Path Connectedness

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**Abstract**

In this paper, some notions of path connectedness and their related theorems are discussed. Path connectedness always implies connectedness but the inverse is not true in general. In particular every open set in Euclidean spaces is connected if and only if it is path connectedness.

**Keywords:** topological space, path, continuous, connectedness, path-connectedness

## Introduction

A path-connected space is a stronger notion of connectedness, requiring the structure of a path. A path from a point x to a point y in the topological space X is a continuous function $\alpha$ from the unit interval $[0,1]$ to X such that $\alpha(0)=x$ and $\alpha(1)=y$. Every path-connected space is connected but the converse is not always true. Every connected subset of the real line, interval, is path connected. Also, open subsets of $\mathbb{R}^n$ and $\mathbb{C}^n$ are connected if and only if they are connected. The connectedness and path-connectedness are the same for finite topological spaces.

## Basic Concept on Path

**Definition.** Let $(X,\tau)$ be a topological space. A ***path*** in X is a continuous function $\alpha:[0,1]\to X$ where $[0,1]$ is equipped with the relative Euclidean topology.

The points $\alpha(0)$ and $\alpha(1)$ are called the initial (or starting) and terminal (or ending) points of $\alpha$. We shall say that $\alpha$ joins x to y or that x is joined to y by $\alpha$.

**Definition.** Let $(X,\tau)$ be a topological space. If $\alpha:[0,1]\to X$ is a path in X then we shall call the path $\overline{\alpha}:[0,1]\to X$ given by $\overline{\alpha}(t)=\alpha(1\text{-}t)$, the ***inverse path*** of $\alpha$. If x is joined to y by means of a path $\alpha$, then y is joined to x by $\overline{\alpha}$.

**Definition.** Given two paths $\alpha,\beta:[0,1]\to X$ with $\alpha(1)=\beta(0)$,

we define the product path $\alpha.\beta:[0,1]\to X$ of $\alpha$ and $\beta$ as

$$\alpha.\beta(t)=\begin{cases}\alpha(2t) & ;\ t\in\left[0,\frac{1}{2}\right]\\ \beta(2t\text{-}1) & ;\ t\in\left[\frac{1}{2},1\right].\end{cases}$$

If $\alpha$ joins x to y and $\beta$ joins y to z, then $\alpha.\beta$ joins x to z.

## Path Connected Topological Spaces

**Definition.** A topological space X is called ***path connected*** if for every two points $x,y\in X$ there exists a path $\alpha:[0,1]\to X$ in X with $\alpha(0)=x$ and $\alpha(1)=y$.

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**Lemma.** Let $(X,\tau_X)$ and $(Y,\tau_Y)$ be two topological spaces and let $f: X \to Y$ be a continuous function. If X is path connected, then $f(X)$ is path connected.

**Proof.** Suppose $x,y \in f(X)$. Let $a,b \in X$ such that $f(a)=x$ and $f(b)=y$.

Since X is path connected, let $\alpha:[0,1] \to X$ be a path such that

We consider,

$$\begin{aligned}(f.\alpha)(0) &= f(\alpha(0)) \\ &= f(a) \\ &= x.\end{aligned}$$

Next,

$$\begin{aligned}(f.\alpha)(1) &= f(\alpha(1)) \\ &= f(b) \\ &= y.\end{aligned}$$

This shows that $f.\alpha$ is a path connecting x and y in $f(X)$. Therefore $f(X)$ is path connected.

**Remark.** We note that if $f: X \to Y$ is continuous and onto, then X is path connected implies that Y is path connected.

**Theorem.** If X is path connected, then it is connected.

**Proof.** Suppose X were not connected. Then we can write

$$X = A \cup B$$

with A and B are two open, disjoint and nonempty sets.

Let $a \in A$ and $b \in B$ be any two points and let $\alpha:[0,1] \to X$ be a path joining a to b. Then the sets $A' = \alpha^{-1}(A)$ and $B' = \alpha^{-1}(B)$ are both open (since $\alpha$ is continuous), nonempty (since $0 \in A'$ and $1 \in B'$) and disjoint (since A and B are disjoint) subsets of $[0,1]$.

This implies that $[0,1]$ is disconnected. It contradicts to theorem. Therefore X must be connected.

**Example.** Euclidean n-space $\mathbb{R}^n$ is path connected and hence also connected.

For given any two points $x,y \in \mathbb{R}^n$, the path $\alpha:[0,1] \to \mathbb{R}^n$ given by $\alpha(t) = x + t(y-x)$.

Thus

$$\begin{aligned}\alpha(0) &= x + 0(y-x) \\ &= x\end{aligned}$$

and

$$\alpha(1) = x + 1(y - x)$$
$$= y.$$

This shows that the path $\alpha$ starts at x and ends at y.

**Example.** For any point $x \in \mathbb{R}^n$ and for any $r > 0$, the ball $B_x(r)$ is path connected, and hence it is also connected.

For every point $y \in B_x(r)$ can be connected to x via the path $\alpha_y : [0,1] \to B_x(r)$, defined by

$$\alpha_y(t) = y + t(x - y).$$

Given any pair of points $y_1, y_2 \in B_x(r)$, the product path $\alpha_{y_1}.\bar{\alpha}_{y_2}$ connects $y_1$ to $y_2$.

Because
$$\alpha_{y_1}(1) = y_1 + 1(x - y_1)$$
$$= x,$$

and
$$\bar{\alpha}_{y_2}(0) = \alpha_{y_2}(1)$$
$$= y_2 + 1(x - y_2)$$
$$= x.$$

Thus

$$\alpha_{y_1}(1) = \bar{\alpha}_{y_2}(0),$$

and the point $y_1$ connects to x via the path $\alpha_{y_1}$ and x connects to $y_2$ via the path $\bar{\alpha}_{y_2}$. So $\alpha_{y_1}.\bar{\alpha}_{y_2}$ is a path, it connects $y_1$ to $y_2$.

**Corollary.** If the Euclidean line $(\mathbb{R}, \tau_{Eu})$ is homeomorphic to Euclidean n-dimensional space $(\mathbb{R}^n, \tau_{Eu})$, then $n = 1$.

**Proof.** Suppose that $f : \mathbb{R} \to \mathbb{R}^n$ is a homeomorphism.

Then

$$f|_{\mathbb{R}\setminus\{0\}} : \mathbb{R} \setminus \{0\} \to \mathbb{R}^n \setminus \{f(0)\}$$

is also a homeomorphism.

However, $\mathbb{R} \setminus \{0\}$ is not an interval and is therefore not connected according to theorem.

On the other hand, we claim that $\mathbb{R}^n \setminus \{f(0)\}$ is path connected, and therefore connected, when $n \geq 2$.

To see this, let $x, y \in \mathbb{R}^n$ be any two points and consider the straight line path

$$\alpha(t) = x + t(y - x).$$

If $f(0)$ does not lie on $\alpha$, then $\alpha$ is a path in $\mathbb{R}^n \setminus \{f(0)\}$ from x to y.

If $f(0)$ does lie on $\alpha$, let $z \in \mathbb{R}^n$ be any point not collinear with x and y.

Consider the paths

$$\beta(t) = x + t(z - x)$$

and

$$\gamma(t) = z + t(y - z)$$

connecting x to z and z to y respectively.

Then $\beta.\gamma$ is a path from x to y in $\mathbb{R}^n \setminus \{f(0)\}$ showing that $\mathbb{R}^n \setminus \{f(0)\}$ is path connected.

Since connectedness is a topological invariant, $\mathbb{R} \setminus \{0\}$ must be connected. It contradicts the assumption that $n \geq 2$. It follows that n must be 1.

**Theorem.** An open subset U of Euclidean space $\mathbb{R}^n$ is connected if and only if it is path connected.

**Proof.** If U is path connected, then U is connected.

We only need to show that if U is connected, then it is also path connected.

Let $x \in U$ be any point and define

$$A = \{y \in U \mid x \text{ and } y \text{ can be joined by a path in } U\},$$

$$B = \{y \in U \mid x \text{ and } y \text{ cannot be joined by a path in } U\}.$$

Clearly $X = A \cup B$ and $A \neq \emptyset$, since $x \in A$.

We will show that both A and B are open subsets of U. Since U is connected, this will imply that $B = \emptyset$ and $A = U$, as desired. To see that A is open, let $y \in A$ be any point and let $\alpha : [0,1] \to U$ be a path joining x to y.

Let $\varepsilon > 0$ be such that $B_y(\varepsilon) \subset U$ for any $z \in B_y(\varepsilon)$. Let

$$\beta_z : [0,1] \to U$$

be the radial path from y to z, i.e.,

$$\beta_z(t) = (1 - t)y + tz.$$

Then $\alpha.\beta_z$ is a path from x to z showing that $B_y(\varepsilon) \subset A$. Since $y \in A$ was arbitrary, we conclude that A is open.

To see that B is open, we proceed similarly. Let $y \in B$ be any point and let $\varepsilon > 0$ be such that

$$B_y(\varepsilon) \subset U.$$

If there were a point

$$z \in B_y(\varepsilon) \cap A,$$

there would have to be a path $\alpha : [0,1] \to U$ from z to x.

Letting $\beta_z$ be as in the previous paragraph, the product path $\beta_z.\alpha$ would be a path from y to x contradiction our choice of $y \in B$.

Therefore $B_y(\varepsilon)$ is contained in B and hence B is open.

**Theorem.** Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces and let $X \times Y$ be given the product topology. Then $X \times Y$ is path connected if and only if each of X and Y are path connected.

**Proof.** We need to show that if X and Y are path connected, then $X \times Y$ is path connected.

Suppose X and Y are path connected. For any two points $(x_1, y_1)$ and $(x_2, y_2) \in (X \times Y)$. Let $\alpha : [0,1] \to X$ be a path in X joining $x_1$ to $x_2$ with $\alpha(0) = x_1$ and $\alpha(1) = x_2$. Let $\beta : [0,1] \to Y$ be a path in Y joining $y_1$ to $y_2$ with $\beta(0) = y_1$ and $\beta(1) = y_2$.

Thus $\alpha \times \beta : [0,1] \to X \times Y$ is a path in $X \times Y$ joining $(x_1, y_1)$ to $(x_2, y_2)$. Therefore $X \times Y$ is path connected.

We also need to show that if $X \times Y$ is path connected, then X and Y are path connected.

Suppose $X \times Y$ is path connected. Since $X \times Y$ is a product topology of X and Y, there are the projection maps $\pi_X$ and $\pi_Y$, they are continuous surjection maps.

By lemma, the continuous images of a path connected space $X \times Y$ under $\pi_X$ and $\pi_Y$, X and Y, are also path connected.

**Lemma.** Let $(X, \tau_X)$ be a topological space and let $Y_i \subset X, i \in I$ be a family of path connected subspaces of X. If $\bigcap_{i \in I} Y_i \neq \varnothing$, then $\bigcup_{i \in I} Y_i$ is a path connected subspace of X.

**Proof.** Let $p \in \bigcap_{i \in I} Y_i$ be any point and for $x \in Y_i$, let $\alpha_x$ be a path in $Y_i$ connecting x to p. Given any $x, y \in \bigcup_{i \in I} Y_i$, the path $\alpha_x.\bar{\alpha}_y$ connects x to y.

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**References**

Joshi, K. D., (1983). *Introduction to General Topology*, Revised Edition, New Age International (P) Limited, New Delhi.

Simmons, G. F., (1963). *Introduction to Topology and Modern Analysis*, International Student Edition, McGRAW-HILL, New York.