Studying Progressive Wave Motion from Mathematical Point of View

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Abstract

In this paper, I introduce two kinds of liquid waves such as long waves and surface waves, especially I focus on the motion of surface wave. Moreover, characteristics of progressive wave and stationary wave are discussed. Finally, I solve the problem by using properties of velocity potential on progressive waves.

Keywords: Equation of motion, Bernoulli's equation

Introduction

The dynamics of wave motion is very important in physical investigations, as wave motion is one of the main modes of transmission of energy. The energy from the sun is transmitted by waves. When some musical instrument is played upon in a room, sound waves spread through the room. If we throw a stone in a pond we find waves in the pond which start from the point of striking of the stone and spread in all directions. Such water waves are also produced by pressure of wind upon the surface of water, by the relative motion of bodies like a ship moving in a sea and by obstructions in the bed of the stream.

Description of a Wave Motion

A wave motion of a liquid acted upon by gravity and having a free surface is a motion in which the elevation of the free surface above some chosen fixed horizontal plane varies. In general, a wave motion may be represented by an equation of the form

(1) $$y = f(x - ct)$$

by taking the axis of x to be horizontal and the axis of y to be vertically upwards.

Here y represents the displacement of a particle situated at x at time t. Increasing t by T and x by cT, we find

R.H.S. of (1) $= f\{x + cT - c(t + T)\} = f(x - ct) =$ L.H.S. of (1).

Thus (1) shows that the wave profile $y = f(x)$ moves with velocity c in the positive x-direction. Likewise, we can show that $y = f(x + ct)$ represents a progressive wave travelling in the negative x-direction with a velocity c. A progressive wave is one in which the surface pattern moves forward. In two dimensional waves the curve in which the free surface meets the plane of motion is called a **wave profile**.

Long waves arise when the depth of the liquid is small compared to the wave length and the disturbance affects the motion of the whole of the liquid. In these waves the vertical acceleration of the liquid is negligible as compared with the horizontal acceleration and the plane of the liquid moves as a whole.

Surface waves occur when the wave length of the oscillations is small compared to the depth of the liquid and hence the disturbance does not extend far below the surface. In these waves the vertical acceleration is appreciable and so it cannot be neglected. Wind waves and surface tension waves are examples of surface waves. Such waves occur in deep and unbounded (in horizontal directions) liquids like lakes and oceans.

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Characteristic of a Wave Motion

Taking the x-axis to be horizontal and the y-axis to be vertically upwards, a motion in which the equation of the vertical section of the free surface at time t is of the form

(2) $$y = a \sin(mx - nt),$$

where a, m, n are constants, is known as a simple harmonic progressive wave.

(2) may also be written as

(3) $$y = a \sin m(x - nt/m).$$

If we increase t by T and x by $(n/m)\times T,$ then we find

R.H.S. of (3) $= a \sin m\left(x + \frac{n}{m}T - \frac{n}{m}(t+T)\right) = a \sin m\left(x - \frac{n}{m}t\right) =$ L.H.S. of (2).

Hence (3) shows that the wave profile $y = a\sin mx$ at time $t = 0$ moves with velocity $\frac{n}{m} = c$ in the positive x-direction, c is called the velocity of propagation of the wave. When $a = 0$, the profile of the liquid is $y = 0$, which is the mean level. The maximum value of y, namely a, is known as the amplitude of the wave.

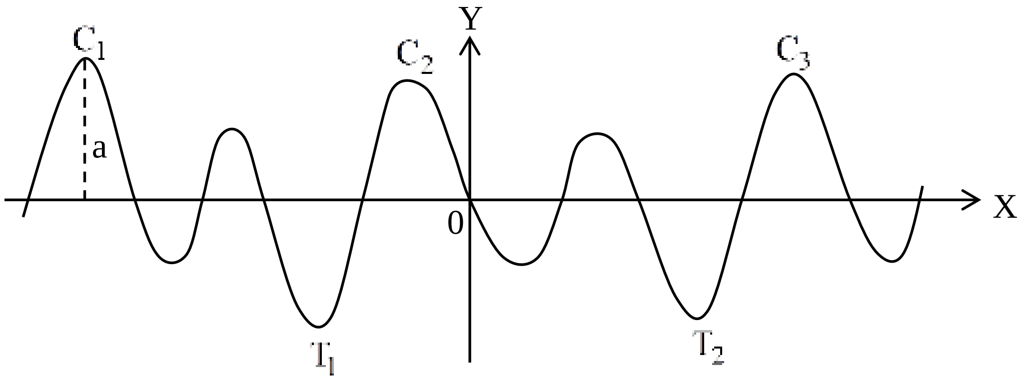

Figure (1)

The points $C_1, C_2, ...,$ of maximum elevation are known as crests and the points $T_1, T_2, ...,$ of maximum depression are known as troughs. The distance between two consecutive crests is known as the wave length and is denoted by λ. Thus $\lambda = (2\pi/m)$. Again the nature of the free surface (2) remains unchanged by replacing t by $t + (2\pi/n)$. The time $T = (2\pi/n)$ is known as the period of the wave. Since $c = (n/m)$ and $\lambda = (2\pi/m)$, we get $T = (\lambda/c)$. The reciprocal of the period is known as the frequency, it denotes the number of oscillations per second, the angle $mx - nt$ is known as the phase angle. If the equation of wave motion may be $y = a \sin(mx - ct + \varepsilon)$ then ε is called the phase of the wave.

General Form of the Superposition of Two Waves

Two simple harmonic progressive waves of the same amplitude, wave length and period travelling in opposite directions are given by the surface elevations

$$y_1 = (a/2)\times\sin(mx - nt) \quad \text{and} \quad y_2 = (a/2)\times\sin(mx + nt).$$

By the principle of superposition, the resulting surface elevation is represented by the equation

$$y = y_1 + y_2, \quad \text{that is} \quad y = a \sin mx \cos nt.$$

A motion of this type is known as stationary wave. At a given value of x the surface of water moves up and down. At any instant the equation represents a sine curve of amplitude $a \cos nt$, which therefore varies between 0 and a. Thus wave of this nature is not propagated. The points of intersection of the curve with the

x-axis are given by $\sin mx = 0$, namely $x = n\pi / m = \lambda n / 2$ where $n = 0, \pm 1, \pm 2, ...$

These points are called **nodes** and the intermediate points where the amplitude is maximum are called **antinodes**.

Again, if $y_3 = a \sin mx \cos nt$ and $y_4 = a \cos mx \sin nt$ may be two stationary waves, the result of superposing these is the elevation

$$y = y_3 \pm y_4, \quad \text{that is} \quad y = a \sin (mx \pm nt).$$

Hence a progressive wave can be regarded as the combination of two systems of stationary waves of the same amplitude, wave length and period, the crests and troughs of one system coinciding with the nodes of the other and their phases differing by a quarter period.

Surface Waves Motion

Surface waves occur at and near the free surface of an unbounded sheet of liquid where the depth is considerable compared to the wave length. For these waves the vertical acceleration is comparable with the horizontal acceleration, and so we consider forces both in horizontal and vertical directions.

Let the x-axis be taken in the undisturbed surface in the direction of propagation of the waves and the y-axis vertically upwards. Taking the motion to be irrotational, incompressible and two dimensional, the velocity potential ϕ exists such that throughout the liquid

(4) $$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

and at a fixed boundary $$\frac{\partial \phi}{\partial x} = 0.$$

The pressure can be obtained from the Bernoulli's equation

(5) $$\frac{p}{\rho} = \frac{\partial \phi}{\partial t} - gy - \frac{q^2}{2} = F(t)\,.$$

Since the surface of equipressure p is constant,

(6) $$\frac{Dp}{Dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} = 0,$$

where u and v are the velocity components on the free surface in x and y directions respectively.

Since, $u = -\dfrac{\partial\phi}{\partial x}$ and $v = -\dfrac{\partial\phi}{\partial y}$,

hence at the free surface the relation (6) becomes

(7) $$\frac{\partial p}{\partial t} - \frac{\partial\phi}{\partial x}\frac{\partial p}{\partial x} - \frac{\partial\phi}{\partial y}\frac{\partial p}{\partial y} = 0.$$

Let the motion be so small that the squares of small quantities may be omitted. So, I omit the higher order term in (5). Again, without loss of generality I may include F(t) in ϕ and hence I will take $F(t) = 0$ in (5). Then, (5) reduces

(8) $$\frac{p}{\rho} = \frac{\partial\phi}{\partial t} - gy.$$

Substituting the value of from (8) in (7),

(9) $$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x\partial t} - \frac{\partial\phi}{\partial y}\left(\frac{\partial^2\phi}{\partial y\partial t} - g\right) = 0,$$

or, omitting the second and third terms which are of the higher order terms,

(10) $$\frac{\partial^2\phi}{\partial t^2} + g\left(\frac{\partial\phi}{\partial y}\right) = 0.$$

Condition (4) must be satisfied at the free surface. If η is the elevation of the free surface at time t,

(11) $$\dot{\eta} = v = -\frac{\partial\phi}{\partial y},$$ which holds at the free surface.

Progressive Waves on the Surface of a Canal

Consider the propagation of simple harmonic waves of the type

(12) $$\eta = a\sin(mx - nt)$$

at the surface of canal of uniform depth h and having parallel vertical walls. Let the free surface be along the x-axis so that equation of the bottom is $y = -h$. Then I must find ϕ satisfying the following boundary conditions

(13) $$\frac{\partial\phi}{\partial y} = 0, \qquad \text{at } y = -h$$

(14) $$\frac{\partial^2\phi}{\partial t^2} + g\left(\frac{\partial\phi}{\partial y}\right) = 0, \qquad \text{at } y = 0$$

(15) $$v = \frac{\partial\eta}{\partial t} = -\frac{\partial\phi}{\partial y} \qquad \text{at } y = 0.$$

$$\frac{\partial\eta}{\partial t} = -an\cos(mx - nt)$$

$$\frac{\partial\phi}{\partial y} = -\frac{\partial\eta}{\partial t} = an\cos\,(mx - nt).$$

By integrating,

(16) $$\phi = f(y)\cos\,(mx - nt)\,.$$

$$\frac{\partial\phi}{\partial x} = -m f(y)\sin\,(mx - nt)$$

$$\frac{\partial^2\phi}{\partial x^2} = -m^2 f(y)\cos\,(mx - nt)$$

$$\frac{\partial^2\phi}{\partial y^2} = f''(y)\cos\,(mx - nt)$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

$$-m^2 f(y)\cos\,(mx - nt) + f''(y)\cos\,(mx - nt) = 0$$

$$-m^2 f(y) + f''(y) = 0$$

$f(y) = Ae^{my} + Be^{-my}$, A, B being arbitrary constants, and hence substitute in (16),

(17) $$\phi = (A e^{my} + B e^{-my})\cos\,(mx - nt)$$

$$\frac{\partial\phi}{\partial y} = (A me^{my} - B me^{-my})\cos\,(mx - nt)$$

substitute in (13),

$$(Am e^{-mh} - Bm e^{mh})\cos\,(mx - nt) = 0$$

$$Ae^{-mh} = Be^{mh} = \frac{D}{2}$$

$$A = \frac{D}{2}e^{mh}\ ,\ \ B = \frac{D}{2}e^{-mh}$$

substitute in (17),

(18) $$\phi = D\cosh m(y + h)\cos\,(mx - nt)$$

$$\frac{\partial\phi}{\partial t} = Dn\cosh m(h + y)\sin\,(mx - nt)$$

$$\frac{\partial^2\phi}{\partial t^2} = -Dn^2\cosh m(h + y)\cos\,(mx - nt)$$

$$\frac{\partial\phi}{\partial y} = Dm \sinh m(h+y) \cos(mx - nt).$$

Substitute in (14),

(19) $$n^2 = gm \tanh mh.$$

Let $c = \frac{n}{m}$ and $\lambda = \frac{2\pi}{m}$ denote the velocity of propagation and the wave length respectively. Then (19) reduces to

$$c^2 = \left(\frac{g}{m}\right)\tanh mh$$

or $$c^2 = \left(\frac{g\lambda}{2\pi}\right)\tanh\left(\frac{2\pi h}{\lambda}\right).$$

I will now determine the constant D of (18) in terms of the amplitude a of the wave. Using (12) and (18), the boundary condition (15) gives $na = mD\sin mh$.

$$\phi = \frac{na}{m}\frac{\cosh m(y+h)}{\sinh mh}\cos(mx - nt)$$

or using (19) $$\phi = \frac{ga}{n}\frac{\cosh m(y+h)}{\cosh mh}\cos(mx - nt).$$

The velocity components of the particles are

$$u = -\frac{\partial\phi}{\partial x} = na\frac{\cosh m(y+h)}{\sinh mh}\sin(mx - nt)$$

$$v = -\frac{\partial\phi}{\partial y} = -na\frac{\sinh m(y+h)}{\sinh mh}\cos(mx - nt).$$

I will now determine the path of the particles. Let (x', y') be the coordinates of a particle relative to its mean position (x, y) such that $|z'| = |x' + iy'|$ is very small. Neglecting the squares of small quantities, for a wave of small elevation the velocities at $z = x + iy$ and $z + z' = (x + x') + i(y + y')$ will be equal.

(20) $$\frac{dx'}{dt} = u = na\frac{\cosh m(y+h)}{\sinh mh}\sin(mx - nt) \text{ and}$$

(21) $$\frac{dy'}{dt} = v = -na\frac{\sinh m(y+h)}{\sinh mh}\cos(mx - nt).$$

Integrating w.r.t ‘t’, (20) and (21) gives

$$x' = a\frac{\cosh m(y+h)}{\sinh mh}\cos(mx - nt),$$

$$y' = a\frac{\sinh m(y+h)}{\sinh mh}\sin(mx-nt).$$

Let $a' = \frac{a\cosh m(y+h)}{\sinh mh}$ and $b' = \frac{a\sinh m(y+h)}{\sinh mh}$.

$$\frac{x'^2}{a'^2}+\frac{y'^2}{b'^2}=1.$$

The path of particle describes the ellipse about its mean position.

Progressive Waves on a Deep Canal

If the depth h of the canal is sufficiently great in comparison with λ for e^{-mh} to be neglected, then in progressive waves on the surface of a canal , $B = 0.$ Thus, I have

(22) $$\phi = Ae^{my}\cos(mx-nt),$$

$$\frac{\partial\phi}{\partial t} = Ane^{my}\sin(mx-nt)$$

$$\frac{\partial^2\phi}{\partial t^2} = -An^2e^{my}\cos(mx-nt)$$

(23) $$\frac{\partial\phi}{\partial y} = Ame^{my}\cos(mx-nt)$$

therefore,

(24) $$n^2 = gm.$$

Let $c = \frac{n}{m}$ and $\lambda = \frac{2\pi}{m}$ denote the velocity of propagation and wave length respectively. Then (24) reduces to

$$c^2 = \frac{g\lambda}{2\pi}.$$

I determine the constant A of (22) in terms of the amplitude a of the wave. Using (12) and (22), the boundary condition gives $na = mA$, so that

$$\phi = \left(\frac{na}{m}\right)e^{my}\cos(mx-nt),$$

or $$\phi = \left(\frac{ga}{m}\right)e^{my}\cos(mx-nt).$$

The velocity components of the particles are

$$u = -\left(\frac{\partial\phi}{\partial x}\right) = nae^{my}\sin(mx-nt)$$

and $$v = -\left(\frac{\partial\phi}{\partial y}\right) = -nae^{my}\cos(mx - nt).$$

By integrating w.r.t. 't',

$$x' = ae^{my}\cos(mx - nt) \text{ and } y' = ae^{my}\sin(mx - nt).$$

Hence the path of the particle is a circle $x'^2 + y'^2 = (ae^{my})^2$, of radius ae^{my}.

Example. I have to consider the ratio of pressure at a depth h.

Consider simple harmonic waves of length λ, at a depth h.

The velocity potential for deep water

$$\phi = \left(\frac{na}{m}\right)e^{my}\cos(mx - nt)$$

(25) $$\frac{\partial\phi}{\partial t} = \left(\frac{an^2}{m}\right)e^{my}\sin(mx - nt).$$

Also the elevation of wave $\eta = a\sin(mx - nt)$ and $c^2 = \frac{n^2}{m^2} = \frac{g}{m}$. So (25) becomes

$$\frac{\partial\phi}{\partial t} = g\eta e^{my}.$$

Pressure equation at any point within the water is

(26) $$\frac{p}{\rho} - \frac{\partial\phi}{\partial t} + gy = c \ (c = \text{constant}).$$

When $y = 0$, $p = 0$, $\frac{\partial\phi}{\partial t} = 0$ so $c = 0$ and hence (25) gives

so $$p = \rho\left(\frac{\partial\phi}{\partial t}\right) - g\rho y \text{ or } p = \rho g\eta e^{my} - g\rho y.$$

Take disturbed pressure p_1.

When $y = -h$, disturbed pressure becomes

$$p_1 = \rho g\eta e^{mh} + g\rho y = \rho gh\left\{1 + \frac{\eta}{h}e^{-mh}\right\}.$$

Next, I will take undisturbed pressure p_2.

When $y = h$, undisturbed pressure becomes

$$p_2 = \rho gh.$$

The ratio of disturbed pressure and undisturbed pressure is

$$p_1 : p_2 = \left(1 + \frac{\eta}{h} e^{-mh}\right) : 1$$

$$p_1 : p_2 = \left(1 + \frac{\eta}{h} e^{-\frac{2\pi h}{\lambda}}\right) : 1, \quad \text{as} \quad m = \frac{2\pi}{\lambda}.$$

Hence simple harmonic waves of length λ are propagated over the surface of deep water, at a point whose depth below the undisturbed surface is h, the pressure at the instants when the disturbed depth of the point is $h + \eta$ bears to the undisturbed pressure at the same point the ratio $1 + \left(\frac{\eta}{h}\right) e^{-\frac{2\pi h}{\lambda}} : 1,$ atmospheric pressure and surface tension being neglected.

Example. I have to find the ship velocity overtakes the waves.

(27) Let the velocity propagation of deep water be $c^2 = \dfrac{g\lambda}{2\pi}$.

Let u be the velocity of the ship. Then, I have

(28) $$\left(\frac{33}{2}\right)(c - u) = \lambda \quad \text{and} \quad 6(c - u) = 220.$$

From (28), $$c - u = \frac{2\lambda}{33} \quad , \quad c - u = \frac{220}{6}$$

$$\frac{2\lambda}{33} = \frac{220}{6}$$

$$\lambda = 605 \text{ ft}.$$

So from (27), $$c = \sqrt{\frac{g\lambda}{2\pi}}$$

$$c = \sqrt{\frac{32 \times 605 \times 7}{2 \times 22}} \text{ ft./sec.}$$

$$c = 55 \text{ ft./sec. approx.}$$

From (28), $$u = \left(c - \frac{110}{3}\right) \text{ft./sec.}$$

$$u = \frac{55}{3} \text{ ft./sec.}$$

Hence the crests of roller which are directly following a ship 220 ft long are observed to overtake it at an interval of 33/2 seconds and it takes a crest 6 seconds to run along the ship.

Results and Conclusion

This research indicates the path of the particle of progressive waves on the surface of a canal and deep canal. In a canal, the path of the particle describes the ellipse about its mean position. Their major axes are horizontal and the lengths of both axes decrease as the depth of the particle increases, the minor axis vanishing at the bottom, degenerating ellipse into a straight line where the particles execute to and fro-motion. In a deep canal, the path of the particle is a circle, the radius decreases with depth of a particle under consideration. For fluid particles on the surface of the liquid, the radius is equal to the amplitude of the wave.

Acknowledgement

I would like to express my heartfelt gratitude to Dr Nilar Myint, and Dr Mar Lar, Pro-Rectors of Hinthada University, for their permission to write this research paper. I would also like to extend my gratitude to Dr Nila Swe, Professor and head, Dr Khin Lay Nyo Nyo, Professor, Department of Mathematics, Hinthada University, for their advice and encouragement to do this research.

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